FRACTIONAL INTEGRATION IN ORLICZ SPACES. I(1)

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Introduction. The present research began when the author tried to generalize classical fractional integration theorems on the interval $(0, 2\pi)$ for spaces $L^p(\log L)^\alpha$. He soon discovered, to his pleasant surprise, that it was easier to work with general Orlicz spaces L_A. The earliest results along this line are presented in §II (Theorems 2.3 and 2.5). They not only suggested to the author what the proper statement of the principal theorem (Theorem 4.7) should be, but also are of interest in themselves. (Indeed because of their elementary nature, the theorems in §II are perhaps the most interesting in the entire paper.)

The notion of "convolution operator" introduced in §III and used most particularly in Lemma 4.2, makes clear that for the type of fractional integration discussed here, little or no role is played by the metric or group theoretic properties of the spaces which support the functions to be "convoluted," i.e., only the measure theoretic properties are used. The author is grateful to Professor E. M. Stein for suggesting to him the notion of "convolution operator."

In §I appears a rather technical definition of "Young's function." On a first reading not much generality will be lost if §I is omitted and if by the words "Young's function" the reader understands a convex monotone increasing, continuous function A defined on $[0, \infty)$ with $A(0) = 0$.

The results of §II answer the following questions. (Let $A(x)$, $B(x)$, $C(x)$ be Young's functions.)

1. If $\int A(|f(x)|)dx$ and $\int B(|g(x)|)dx$ are finite, is $\int C(|f(x)g(x)|)dx$ finite?

(We shall see that a sufficient and with a minor qualification, necessary condition is that $A^{-1}(x)B^{-1}(x) \leqq C^{-1}(x)$ for all $x \geqq 0$.)

2. If $\int A(|f(x)|)dx$ and $\int B(|g(x)|)dx$ are finite and

$$h(x) = \int f(x-t)g(t)\,dt$$

is $\int C(|h(x)|)\,dx$ finite?

(A sufficient condition is that $A^{-1}(x)B^{-1}(x) \leqq xC^{-1}(x)$.)

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These questions are discussed further in §VI. (Some readers may wish to omit entirely §§III, IV and V which discuss the fractional integration theory.)

In §III the Banach spaces M_A which are closely related to L_A are introduced. The idea for these spaces is essentially due to G. G. Lorentz [4].

We shall not discuss here why the convolution of a function g, with a kernel in M_A is called a "fractional integration" of g. Those interested in this question may refer to Zygmund [10, Vol. II, pp. 133-142] or to Hardy and Littlewood [2].

In §IV is presented the principal result on fractional integration. The reader will observe that there is a very close connection between the condition relating the Young's functions A, B, C in Lemma 4.2 and the condition in Theorem 2.5. Theorem 4.7 is in some sense the principal result of this paper.

In §V the results in §IV are extended to some rather special cases. §V may be of more interest to the specialist than to the casual reader.

§VI and VII are in the nature of appendices mostly to §II.

The necessity to develop tools and notations for attacking the theorems has led to greater length than the author would have desired, but it has had the virtue of making this paper largely self-contained.

I. Young's functions and Orlicz spaces.

DEFINITION 1.1. *$A(x)$ is a Young's function if either*

Case 1. *$A(x)$ is a convex, nondecreasing, finite-valued function which is not identically* 0 *on* $[0, \infty)$ *and* $A(0) = 0$, *or*

Case 2. *There is a number $x_1 > 0$ such that A is convex, nondecreasing and finite valued on $[0, x_1]$, $A(0) = 0$, and for $x > x_1$, $A(x) = \infty$, or*

Case 3. *There is a number $x_1 > 0$ such that A is convex, nondecreasing and finite valued on $[0, x_1)$, $A(0) = 0$, $\lim_{x \to x_1^-} A(x) = \infty$, and for $x \geqq x_1$, $A(x) = \infty$, or*

Case 4. *$A(0) = 0$ and for $x > 0$, $A(x) = \infty$. We shall call this the "trivial" Young's function.*

DEFINITION 1.2. *If A is a Young's function then A^{-1} is defined for $0 \leqq y \leqq \infty$ by*

$$A^{-1}(y) = \inf\{x : A(x) > y\}$$

where $\inf \phi = \infty$.

REMARKS.

1°. For $0 \leqq x < \infty$, $A(x) = \sup\{y : A^{-1}(y) < x\}$, where $\sup \phi = 0$.

2°. The domain of A is $[0, \infty)$; the domain of A^{-1} is $[0, \infty]$.

3°. In all cases $A^{-1}(\infty) = \infty$.

4°. A is continuous to the left while A^{-1} is continuous to the right.

5°. By allowing A to jump to ∞ at x_1 we may include L^∞ as an Orlicz space.

6°. The following useful inequalities are valid. The reader may easily verify them for himself.

Property 1.3. *If A is a Young's function and if $0 \leqq x < \infty$ then*

$$x \leqq A^{-1}(A(x))$$

and

$$A(A^{-1}(x)) \leqq x.$$

Definition 1.4. *The Orlicz space $L_A = L_A(X)$ is the set of all (real- or complex-valued) measurable functions on the measure space (X, μ) for which there exists a number $K > 0$ such that*

$$\int_X A\left(\frac{|f(x)|}{K}\right) d\mu \leqq 1.$$

The norm $\|f\|_A$ is defined as the inf *of such K.*

Definition 1.5. *Given a Young's function A, the Young's complement $\bar{A}$ is defined for $0 \leqq x < \infty$ by*

$$\bar{A}(x) = \sup_{0 \leqq y < \infty} (xy - A(y)).$$

The following is well known and may easily be verified by the reader.

Property 1.6. *If A is a nontrivial Young's function, then $\bar{A}$ is a nontrivial Young's function. If $B = \bar{A}$ then $A = \bar{B}$. For all $0 \leqq x < \infty$,*

$$x \leqq A^{-1}(x)\bar{A}^{-1}(x) \leqq 2x.$$

For all $0 \leqq x < \infty$, $0 \leqq y < \infty$,

$$xy \leqq A(x) + \bar{A}(y) \qquad (\textit{Young's inequality}).$$

Using Young's inequality and Definition 1.4, G. Weiss [8] has shown the following generalization of Hölder's inequality:

$$\int_X |f(x)g(x)|\, d\mu \leqq 2\|f\|_A\|g\|_{\bar{A}}. \tag{1.7}$$

II. Hölder's inequality and Young's theorem. One immediate consequence of Young's inequality is:

If p and q are positive numbers such that $1/p + 1/q = 1$ then

$$xy \leqq \frac{x^p}{p} + \frac{y^q}{q}.$$

From this we easily derive Hölder's inequality. An immediate corollary of Hölder's inequality is:

If $f \in L^p$ and $g \in L^q$ on a measure space (X, μ) and if $1/r = 1/p + 1/q$ then

$$\left[\int |f(x)g(x)|^r \, d\mu\right]^{1/r} \leqq \|f\|_p \|g\|_q.$$

We shall generalize this for Orlicz spaces by means of the inequality of Lemma 2.1 which is a generalization of Young's inequality.

LEMMA 2.1 (FIRST GENERALIZED YOUNG'S INEQUALITY). *If A, B, C, are extended real-valued, non-negative, nondecreasing, left continuous functions defined on $[0, \infty)$, if for $0 \leqq x \leqq \infty$, $A^{-1}(x) = \inf\{y : A(y) > x\}$ $(\inf \phi = \infty)$, similarly define B^{-1}, C^{-1}, then if for all $0 \leqq x < \infty$*

$$A^{-1}(x)B^{-1}(x) \leqq C^{-1}(x),$$

then for all $0 \leqq x < \infty$, $0 \leqq y < \infty$

$$C(xy) \leqq A(x) + B(y).$$

Proof. It follows from the definition of A^{-1} that for all $x \geqq 0$, $A(A^{-1}(x)) \leqq x \leqq A^{-1}(A(x))$.

Given any $x \geqq 0$, $y \geqq 0$, either $A(x) \leqq B(y)$ or $A(x) > B(y)$.

If $A(x) \leqq B(y)$ then

$$xy \leqq A^{-1}(A(x))B^{-1}(B(y)) \leqq A^{-1}(B(y))B^{-1}(B(y))$$
$$\leqq C^{-1}(B(y)).$$
$$C(xy) \leqq C(C^{-1}(B(y))) \leqq B(y).$$

If $A(x) > B(y)$, a similar argument shows that $C(xy) \leqq A(x)$.

Therefore,

$$C(xy) \leqq \max(A(x), B(y)) \leqq A(x) + B(y).$$

THEOREM 2.2. *If A, B, C are extended real-valued non-negative, nondecreasing, left continuous functions defined on $[0, \infty)$, if*

$$A^{-1}(x) = \inf\{y : A(y) > x\},$$

similarly for B^{-1}, C^{-1}, and if

$$A^{-1}(x)B^{-1}(x) \leqq C^{-1}(x),$$

then

$$\int C(|f(x)g(x)|) \, d\mu \leqq \int A(|f(x)|) \, d\mu + \int B(|g(x)|) \, d\mu.$$

Proof. We integrate the following inequality which is an immediate consequence of Lemma 2.1.

$$C(|f(x)g(x)|) \leqq A(|f(x)|) + B(|g(x)|).$$

THEOREM 2.3 (GENERALIZED HÖLDER'S INEQUALITY). *If A, B, C are Young's functions such that*

$$A^{-1}(x)B^{-1}(x) \leqq C^{-1}(x)$$

and if $f \in L_A$, $g \in L_B$ *on a measure space* (X, μ), *then the product* $h(x) = f(x)g(x)$ *is in* L_C *and*

$$\|h\|_C \leqq 2\|f\|_A \|g\|_B.$$

Proof. Let $\epsilon > 0$; without loss of generality assume $\|f\|_A = 1 = \|g\|_B$. Then, using Theorem 2.2 and the convexity of $C(x)$,

$$\begin{aligned}
\int C\left(\frac{|h(x)|}{2(1+\epsilon)^2}\right) d\mu &\leqq \frac{1}{2}\int C\left(\frac{|f(x)|}{1+\epsilon}\frac{|g(x)|}{1+\epsilon}\right) d\mu \\
&\leqq \frac{1}{2}\int A\left(\frac{|f(x)|}{1+\epsilon}\right) d\mu + \frac{1}{2}\int B\left(\frac{|g(x)|}{1+\epsilon}\right) d\mu \\
&\leqq \frac{1}{2} + \frac{1}{2} = 1. \\
\|h\|_C &\leqq 2(1+\epsilon)^2
\end{aligned}$$

and the theorem follows by letting $\epsilon \to 0$.

We shall comment further on this theorem in §VI.

A well-known theorem of Young which we shall generalize in Theorem 2.5 is the following:

THEOREM (W. H. YOUNG). *If* $f \in L^p, g \in L^q$ *on the real line and if* $1/p + 1/q \geqq 1$, *then their convolution*

$$h(x) = \int f(x-t)g(t)\,dt$$

is in L^r *where* $1/p + 1/q = 1/r$ *and* $\|h\|_r \leqq \|f\|_p \|g\|_q$.

LEMMA 2.4 (SECOND GENERALIZED YOUNG'S INEQUALITY). *If* A, B, *are Young's functions such that for* $x \geqq 0$,

$$A^{-1}(x)B^{-1}(x) \leqq xC^{-1}(x),$$

then for $x \geqq 0, y \geqq 0$,

$$xy \leqq A(x)C^{-1}(B(y)) + B(y)C^{-1}(A(x)).$$

Proof. Given $x \geqq 0, y \geqq 0$, either $A(x) \leqq B(y)$ or $A(x) > B(y)$.
If $A(x) \leqq B(y)$,

$$xB^{-1}(A(x)) \leqq A^{-1}(A(x))B^{-1}(A(x)) \leqq A(x)C^{-1}(A(x)),$$

$$x \leqq \frac{A(x)}{B^{-1}(A(x))}C^{-1}(A(x)).$$

But $u/B^{-1}(u)$ is a nondecreasing function so that

$$\frac{A(x)}{B^{-1}(A(x))} \leqq \frac{B(y)}{B^{-1}(B(y))} \leqq \frac{B(y)}{y}.$$

Thus $xy \leqq B(y)C^{-1}(A(x))$.

If $A(x) > B(y)$ a similar argument shows that $xy \leqq A(x)C^{-1}(B(y))$. In either case

$$xy \leqq \max\left(A(x)C^{-1}(B(y)), B(y)C^{-1}(A(x))\right)$$
$$\leqq A(x)C^{-1}(B(y)) + B(y)C^{-1}(A(x)).$$

THEOREM 2.5. *Suppose A, B, C are Young's functions which satisfy, for $x \geqq 0$,*

$$A^{-1}(x)B^{-1}(x) \leqq xC^{-1}(x),$$

and that $f \in L_A$, $g \in L_B$ on a locally compact unimodular topological group (G, μ), where μ is the Haar measure. Then their convolution (writing dt instead of $d\mu$),

$$h(x) = \int_G f(t)g(t^{-1}x)\,dt$$

is in $L_C(G, \mu)$ and

$$\|h\|_C \leqq 2\|f\|_A\|g\|_B.$$

Proof. Let $\epsilon > 0$. Without loss of generality we may suppose $\|f\|_A = 1 = \|g\|_B$.

By Lemma 2.4 and Jensen's inequality [10, Vol. I, p. 21, (10.1)]:

$$\int C\left(\frac{|h(x)|}{2(1+\epsilon)^2}\right) dx \leqq \int C\left(\frac{1}{2}\int \frac{|f(t)|}{1+\epsilon}\frac{|g(t^{-1}x)|}{1+\epsilon}\,dt\right) dx$$
$$\leqq \int C\left[\frac{1}{2}\int A\left(\frac{|f(t)|}{1+\epsilon}\right) C^{-1}\left(B\left(\frac{|g(t^{-1}x)|}{1+\epsilon}\right)\right) dt\right.$$
$$\left. + \frac{1}{2}\int B\left(\frac{|g(t^{-1}x)|}{1+\epsilon}\right) C^{-1}\left(A\left(\frac{|f(t)|}{1+\epsilon}\right)\right) dt\right] dx$$
$$\leqq \frac{1}{2}\int C\left[\int A\left(\frac{|f(t)|}{1+\epsilon}\right) C^{-1}\left(B\left(\frac{|g(t^{-1}x)|}{1+\epsilon}\right)\right) dt\right] dx$$
$$+ \frac{1}{2}\int C\left[\int B\left(\frac{|g(t^{-1}x)|}{1+\epsilon}\right) C^{-1}\left(A\left(\frac{|f(t)|}{1+\epsilon}\right)\right) dt\right] dx$$
$$= I + J.$$

We use Jensen's inequality [10, Vol. I, p. 24, (10.8)], observing that

$$\int A\left(\frac{|f(t)|}{1+\epsilon}\right) dt \leqq 1.$$

$$I=\frac{1}{2}\int C\left[\int A\left(\frac{|f(t)|}{1+\epsilon}\right) C^{-1}\left(B\left(\frac{|g(t^{-1}x)|}{1+\epsilon}\right)\right) dt\right] dx$$

$$\leqq \frac{1}{2}\int C\left[\frac{\int A\left(\frac{|f(t)|}{1+\epsilon}\right) C^{-1}\left(B\left(\frac{|g(t^{-1}x)|}{1+\epsilon}\right)\right) dt}{\int A\left(\frac{|f(t)|}{1+\epsilon}\right) dt}\right] dx$$

$$\leqq \frac{1}{2}\int \frac{\int A\left(\frac{|f(t)|}{1+\epsilon}\right) C\left(C^{-1}\left(B\left(\frac{|g(t^{-1}x)|}{1+\epsilon}\right)\right)\right) dt}{\int A\left(\frac{|f(t)|}{1+\epsilon}\right) dt} dx$$

$$\leqq \frac{1}{2\int A\left(\frac{|f(t)|}{1+\epsilon}\right) dt}\int\int A\left(\frac{|f(t)|}{1+\epsilon}\right) B\left(\frac{|g(t^{-1}x)|}{1+\epsilon}\right) dtdx$$

$$= \frac{1}{2\int A\left(\frac{|f(t)|}{1+\epsilon}\right) dt}\int A\left(\frac{|f(t)|}{1+\epsilon}\right)\left[\int B\left(\frac{|g(t^{-1}x)|}{1+\epsilon}\right) dx\right] dt$$

$$\leqq \frac{1}{2\int A\left(\frac{|f(t)|}{1+\epsilon}\right) dt}\int A\left(\frac{|f(t)|}{1+\epsilon}\right) dt = \frac{1}{2}.$$

Similarly, $J \leqq 1/2$, $I + J \leqq 1$,

$$||h||_C \leqq 2(1+\epsilon)^2.$$

The theorem follows by letting ϵ tend to zero.

Theorem 2.5 is close to the "fractional integration" theorem we are looking for. Roughly speaking, we can enlarge the class L_A to a class of functions which we shall call M_A and still preserve in a slightly altered form the conclusion of Theorem 2.5. Before we give this result in §IV we shall need some notation which we develop in §III.

III. **The Banach space M_A. Convolution operators.** If f is a complex- (or real-) valued measurable function on a measure space (X, μ) then for $y \geqq 0$, $m(f,y) = \mu(E_y)$ where $E_y = \{x : |f(x)| > y\}$. $m(f,y)$ is a monotone nonincreasing function which takes the non-negative reals into $[0, \infty]$. We may form its inverse f^*. For $x \geqq 0$,

$$f^*(x) = \inf\{y : m(f,y) \leqq x\}.$$

f^* is called the nonincreasing rearrangement of $|f|$ onto the positive reals. f^* is equimeasurable with $|f|$.

$$\int |f(x)|\, d\mu = \int_0^\infty f^*(t)\, dt = \int_0^\infty m(f,y)\, dy.$$

The integral mean f^{**} of f^* will be useful to us. For $x > 0$,

$$f^{**}(x) = \frac{1}{x}\int_0^x f^*(t)\, dt,$$

$$f^{**}(0) = f^*(0).$$

It is clear that

$$xf^{**}(x) = \int_0^x f^*(t)\, dt = xf^*(x) + \int_{f^*(x)}^\infty m(f,y)\, dy.$$

Lemma. *If f and g are measurable functions on (X,μ) and if $h = f + g$ then for $x \geqq 0$,*

$$h^{**}(x) \leqq f^{**}(x) + g^{**}(x).$$

Proof. Assume that (X,μ) is atom free, then

$$xh^{**}(x) = \sup \int_E |h(t)|\, d\mu,$$

the supremum taken over sets E such that $\mu(E) = x$. But

$$\int_E |h(t)|\, d\mu \leqq \int_E |f(t)|\, d\mu + \int_E |g(t)|\, d\mu$$

$$\leqq xf^{**}(x) + xg^{**}(x).$$

If (X,μ) has atoms the proof is more complicated but nonetheless we leave it to the reader.

If A is a Young's function and $a(x) = A'(x)$ then

$$\int A(|f(x)|)\, d\mu = \int_0^\infty m(A(|f|),y)\, dy$$

$$= \int_0^\infty m(|f|, A^{-1}(y))\, dy = \int_0^\infty m(f,u)\, dA(u)$$

$$= \int_0^\infty a(u)m(f,u)\, du,$$

where we have made the substitution $y = A(u)$.

Definition. *$f \in M_A$ on (X,μ) if and only if there is a positive number K so large that*

$$f^{**}(x) \leqq KA^{-1}\left(\frac{1}{x}\right).$$

$\|f\|_{M_A}$ is defined as the infimum of all such K.

LEMMA. $\|\cdot\|_{M_A}$ *is a norm.*

Proof. Homogeneity and vanishing only on the zero function are obvious. To show the triangle inequality suppose that $g=f_1+f_2$, then for all $x\leqq 0$,

$$\begin{aligned} g^{**}(x) &\leqq f_1^{**}(x)+f_2^{**}(x)\\ &\leqq \|f_1\|_{M_A}A^{-1}\left(\frac{1}{x}\right)+\|f_2\|_{M_A}A^{-1}\left(\frac{1}{x}\right)\\ &=(\|f_1\|_{M_A}+\|f_2\|_{M_A})A^{-1}\left(\frac{1}{x}\right),\end{aligned}$$

so that

$$\|g\|_{M_A}\leqq\|f_1\|_{M_A}+\|f_2\|_{M_A}.$$

LEMMA 3.1. *If* $f\in L_A$ *then* $f\in M_A$ *and* $\|f\|_{M_A}\leqq\|f\|_A$.

Proof. Let $\epsilon>0$; without loss of generality let $\|f\|_A=1$. By Jensen's inequality [10, Vol. I, p. 24, (10.8)],

$$\begin{aligned} A\left(\frac{f^{**}(y)}{1+\epsilon}\right) &= A\left[\frac{\int_0^y \frac{f^*(t)}{1+\epsilon}dt}{\int_0^y dt}\right]\\ &\leqq \frac{1}{y}\int_0^y A\left(\frac{f^*(t)}{1+\epsilon}\right)dt\leqq\frac{1}{y}\int_0^\infty A\left(\frac{f^*(t)}{1+\epsilon}\right)dt\leqq\frac{1}{y},\\ \frac{f^{**}(y)}{1+\epsilon} &\leqq A^{-1}\left(A\left(\frac{f^{**}(y)}{1+\epsilon}\right)\right)\leqq A^{-1}\left(\frac{1}{y}\right).\end{aligned}$$

The lemma follows by letting ϵ tend to zero.

DEFINITION. *Let* $(X,\mu),(\bar{X},\bar{\mu})$ *and* (Y,ν) *be three (not necessarily distinct) measure spaces and* T *a bilinear operator taking measurable functions on* X *and* $\bar{X}$ *into measurable functions on* Y. *Suppose further that*

$$\begin{aligned} &\|T(f,g)\|_1\leqq\|f\|_1\|g\|_1,\\ (3.2)\qquad &\|T(f,g)\|_\infty\leqq\|f\|_1\|g\|_\infty,\\ &\|T(f,g)\|_\infty\leqq\|f\|_\infty\|g\|_1.\end{aligned}$$

T is called a "convolution operator."

The definition of a convolution operator is to be understood as defining $T(f,g)$ only in the case the existence is forced by equations (3.2), that is, in case one of the functions, say g, belongs to L and the other can be split into a sum, $f=f_1+f_2$, with $f_1\in L, f_2\in L^\infty$. In Lemma 4.1 we shall see that in certain cases it is possible to extend the "convolution operator" in a natural way.

EXAMPLE 3.3. *If (G,μ) is a locally compact unimodular topological group and*

$$h(x) = T(f,g) = \int_G f(t)g(t^{-1}x)\, d\mu(t),$$

then T is a convolution operator.

We remark that in establishing the last two inequalities of (3.2) we need both the right and the left invariance of μ.

EXAMPLE 3.4. *Ordinary convolution on Euclidean n-space is an example of a convolution operator.*

IV. The fractional integration theorem.

LEMMA 4.1. *If $\int_0^\infty f^*(t)g^*(t)\,dt$ is finite then the convolution operator T may be extended so that $T(f,g)$ is defined. Moreover,*

$$\|T(f,g)\|_\infty \leqq \int_0^\infty f^*(t)g^*(t)\,dt. \tag{4.1}$$

Proof. We shall assume throughout the proof that f,g are functions such that $\int_0^\infty f^*(t)g^*(t)\,dt < \infty$. We begin by proving (4.1) in a series of five steps where $T(f,g)$ is already known to exist. In step six we shall extend T.

1°. $|g(x)| = b\chi_E(x), \chi_E$ a characteristic function.

Let $k = \text{mes } E = \|\chi_E\|_1$.

Let

$$f_u(x) = \begin{cases} f(x) & \text{if } |f(x)| \leqq u, \\ u \operatorname{sgn} f(x) & \text{if } |f(x)| > u. \end{cases}$$

Define f^u by the equation: $f(x) = f^u(x) + f_u(x)$. It follows that

$$\|f_u\|_1 = \int_0^u m(f,y)\,dy,$$

and

$$\|f^u\|_1 = \int_u^\infty m(f,y)\,dy,$$

$$h = T(f,g) = T(f^u,g) + T(f_u,g) = h_1 + h_2,$$

$$\|h_1\|_\infty \leqq \|f^u\|_1\|g\|_\infty = \int_u^\infty m(f,y)\,dy \cdot b,$$

$$\|h_2\|_\infty \leqq \|f_u\|_\infty\|g\|_1 \leqq ubk.$$

If we set $u = f^*(k)$,

$$\|T(f,g)\|_\infty \leqq \|h_1\|_\infty + \|h_2\|_\infty$$

$$\leqq bkf^*(k) + b\int_{f^*(k)}^\infty m(f,y)\,dy = bkf^{**}(k)$$

$$= \int_0^k bf^*(t)\,dt = \int_0^\infty f^*(t)g^*(t)\,dt.$$

2°. If $|g(x)|$ is a simple function, let $0 = y_0 < y_1 < \cdots < y_n$ be the values of $|g|$.

Let

$$g_k(x) = \begin{cases} 0 & \text{if } |g(x)| < y_k, \\ g(x) - y_{k-1}\,\mathrm{sgn}\,g(x) & \text{if } |g(x)| = y_k, \\ (y_k - y_{k-1})\,\mathrm{sgn}\,g(x) & \text{if } |g(x)| > y_k, \end{cases}$$

$$g(x) = \sum_{k=1}^n g_k(x).$$

It is easily seen that

$$g^*(t) = \sum_{k=1}^n g_k^*(t).$$

$$\|T(f,g)\|_\infty \leqq \sum_{k=1}^n \|T(f,g_k)\|_\infty$$

$$\leqq \sum_{k=1}^n \int_0^\infty f^*(t)g_k^*(t)\,dt = \int_0^\infty f^*(t)g^*(t)\,dt.$$

3°. If $f \in L$ and $g \in L$. Given $x > 0$ and $\epsilon > 0$, let $\phi_{x,\epsilon}(t)$ be a simple function such that $|\phi(t)| \leqq |g(t)|$ and such that $\|g - \phi\|_1 < \epsilon x$.

$$h = T(f,g) = T(f,\phi) + T(f,g-\phi) = h_1 + h_2.$$

$$h_1^{**}(x) \leqq h_1^{**}(0) = \|h_1\|_\infty \leqq \int_0^\infty f^*(t)\phi^*(t)\,dt$$

$$\leqq \int_0^\infty f^*(t)g^*(t)\,dt.$$

$$xh_2^{**}(x) \leqq \|h_2\|_1 \leqq \|f\|_1\|g - \phi\|_1 = \|f\|_1\epsilon x.$$

$$h^{**}(x) \leqq h_1^{**}(x) + h_2^{**}(x) \leqq \int_0^\infty f^*(t)g^*(t)\,dt + \|f\|_1\epsilon.$$

Let ϵ tend to zero.

$$h^{**}(x) \leqq \int_0^\infty f^*(t)g^*(t)\,dt.$$

But this is true for any $x > 0$.

$$\|h\|_\infty = h^{**}(0) = \lim_{x \to 0} h^{**}(x) \leqq \int_0^\infty f^*(t)g^*(t)\,dt.$$

4°. If $\lim_{x\to\infty} f^*(x) = 0$ and $g \in L$. Given $\epsilon > 0$, there is an x so large that $f^*(x) < \epsilon$. This means there is a set E of measure less than or equal x such that outside $E, |f|$ is less than ϵ. Let $f = f_1 + f_2$ where f_1 is f restricted to E.

$$T(f,g) = T(f_1,g) + T(f_2,g).$$

$$\|T(f_1,g)\|_\infty \leqq \int_0^\infty f_1^*(t)g^*(t)\,dt \leqq \int_0^\infty f^*(t)g^*(t)\,dt.$$

$$\|T(f_2,g)\|_\infty \leqq \|f_2\|_\infty \|g\|_1 \leqq \epsilon \|g\|_1.$$

$$\|T(f,g)\|_\infty \leqq \int_0^\infty f^*(t)g^*(t)\,dt + \epsilon\|g\|_1.$$

Let ϵ tend to zero.

5°. If f is arbitrary and $g \in L$. Let $a = \lim_{x\to\infty} f^*(x)$,

$$f_1(x) = a \operatorname{sgn} f(x),$$

$$f_2(x) = f(x) - f_1(x),$$

$$T(f,g) = T(f_1,g) + T(f_2,g).$$

$$\|T(f_1,g)\|_\infty \leqq \|f_1\|_\infty \|g\|_1 = a\int_0^\infty g^*(t)\,dt.$$

$$\|T(f_2,g)\|_\infty \leqq \int_0^\infty f_2^*(t)g^*(t)\,dt.$$

$$\|T(f,g)\|_\infty \leqq \int_0^\infty [a + f_2^*(t)]g^*(t)\,dt$$

$$= \int_0^\infty f^*(t)g^*(t)\,dt.$$

6°. If $f \notin L$, $g \notin L$ but $\int_0^\infty f^*(t)g^*(t)\,dt < \infty$ we may extend T so that $T(f,g)$ has a meaning. If $\lim_{t\to\infty} f^*(t) = a > 0$, then the finiteness of the above integral would imply $g \in L$. So both $\lim_{t\to\infty} f^*(t) = 0$ and $\lim_{t\to\infty} g^*(t) = 0$. Let E_n be an increasing sequence of sets of finite measure whose union is $E = \{x : g(x) \neq 0\}$. Let $g_n(x) = g(x)$ if $x \in E_n, g_n(x) = 0$ if $x \notin E_n$. Then the operator $V(g_n) = T(f,g_n)$ is a bounded operator taking the Banach space whose norm is given by $\int_0^\infty f^*(t)g_n^*(t)\,dt$ into L^∞. Moreover, g_n is a Cauchy sequence and so $V(g_n)$ converges in L^∞. By definition $T(f,g) = \lim_{n\to\infty} T(f,g_n)$, limit in the L^∞ norm. Since

$$\|T(f,g_n)\|_\infty \leqq \int_0^\infty f^*(t)g_n^*(t)\,dt \leqq \int_0^\infty f^*(t)g^*(t)\,dt,$$

it is clear that

$$\|T(f,g)\|_\infty \leqq \int_0^\infty f^*(t)g^*(t)\,dt.$$

If we had decided instead to extend T by choosing an increasing sequence of sets of finite measure D_m whose union was the support of $f, f_m = f$ restricted to D_m, we would have arrived at the same value for $T(f,g)$.

$$T(f_m,g) - T(f,g_n) = T(f_m,g-g_n) - T(f-f_m,g_n).$$

Let $r_m = f - f_m$, $\lim_{m\to\infty} r_m(x) = 0$; moreover, the convergence is monotone.

$$\|T(f-f_m,g_n)\|_\infty \leqq \int_0^\infty r_m^*(t)g_n^*(t)\,dt \leqq \int_0^\infty r_m^*(t)g^*(t)\,dt$$

and the last integral tends to zero by the Lebesgue monotone convergence theorem. Similarly, $\lim_{n\to\infty}\|T(f_m,g-g_n)\|_\infty = 0$ uniformly in m. Thus, $\lim_{m\to\infty} T(f_m,g) = \lim_{n\to\infty} T(f,g_n)$, lim in L^∞ norm.

Corollary. *Necessary and sufficient conditions that a bilinear operator be a convolution operator are*

$$\|T(f,g)\|_1 \leqq \|f\|_1\|g\|_1,$$

$$\|T(f,g)\|_\infty \leqq \int_0^\infty f^*(t)g^*(t)\,dt.$$

Remark. $T(f,g)$ is defined if and only if the three integrals are finite: $\int_0^1 f^*(t)\,dt$, $\int_0^1 g^*(t)\,dt$, $\int_1^\infty f^*(t)g^*(t)\,dt$.

Lemma 4.2. *If A, B are Young's functions such that*

$$\int_0^1 \frac{A^{-1}(t)B^{-1}(t)}{t^2}\,dt < \infty$$

and

$$C^{-1}(x) = \int_0^x \frac{A^{-1}(t)B^{-1}(t)}{t^2}\,dt,$$

then C is a Young's function.

If $f \in M_A$, $g \in L_B$, $\|f\|_{M_A} \leqq 1$, $\|g\|_B \leqq 1$, and $h = T(f,g)$ where T is a convolution operator then

$$m\left(\frac{h}{3+\theta}, z\right) \leqq \frac{1}{\theta B^{-1}(C(z))}\int_{B^{-1}(C(z))}^\infty m(g,y)\,dy \tag{4.2}$$

where θ is any positive constant.

Proof. Since $A^{-1}(t)/t, B^{-1}(t)/t$ are nonincreasing it follows that $C^{-1}(x)$ is concave and $C^{-1}(0) = 0$. Therefore C is a Young's function. Furthermore,

$$C^{-1}(x) = \int_0^x \frac{A^{-1}(t)}{t} \frac{B^{-1}(t)}{t} dt \geqq x \frac{A^{-1}(x)}{x} \frac{B^{-1}(x)}{x},$$

so that

$$A^{-1}(x)B^{-1}(x) \leqq xC^{-1}(x).$$

As in the proof of 1° in Lemma 4.1, we let

$$f = f_u + f^u, \qquad g = g_w + g^w,$$

where $u = f^*(s)$ and $w = B^{-1}(1/s)$.

$$\begin{aligned} h = T(f,g) &= T(f_u + f^u, g_w + g^w) \\ &= T(f_u, g_w) + T(f_u, g^w) + T(f^u, g_w) + T(f^u, g^w) \\ &= h_1 + h_2 + h_3 + h_4. \end{aligned}$$

By Lemma 4.1,

$$\begin{aligned} \|h_1\|_\infty = \|T(f_u, g_w)\|_\infty &\leqq \int_0^\infty f_u^*(t) g_w^*(t)\, dt \\ &= \int_0^s + \int_s^\infty \leqq uws + \int_s^\infty f^*(t) g^*(t)\, dt \\ &\leqq sf^*(s)B^{-1}\left(\frac{1}{s}\right) + \int_s^\infty A^{-1}\left(\frac{1}{t}\right) B^{-1}\left(\frac{1}{t}\right) dt \\ &= sf^*(s)B^{-1}\left(\frac{1}{s}\right) + \int_0^{1/s} \frac{A^{-1}(y)B^{-1}(y)}{y^2}\, dy \\ &= sf^*(s)B^{-1}\left(\frac{1}{s}\right) + C^{-1}\left(\frac{1}{s}\right). \end{aligned}$$

$$\begin{aligned} \|h_2\|_\infty = \|T(f_u, g^w)\|_\infty &\leqq \|f_u\|_\infty \|g^w\|_1 \\ &\leqq u \int_w^\infty m(g,y)\, dy = f^*(s) \int_{B^{-1}(1/s)}^\infty m(g,y)\, dy \\ &\leqq f^*(s) \int_{g^*(s)}^\infty m(g,y)\, dy \leqq f^*(s) s g^{**}(s) \\ &\leqq sA^{-1}\left(\frac{1}{s}\right) B^{-1}\left(\frac{1}{s}\right) \leqq C^{-1}\left(\frac{1}{s}\right). \end{aligned}$$

$$\begin{aligned} \|h_3\|_\infty = \|T(f^u, g_w)\|_\infty &\leqq \|f^u\|_1 \|g_w\|_\infty \\ &\leqq \int_u^\infty m(f,y)\, dy \cdot w = \int_{f^*(s)}^\infty m(f,y)\, dy \cdot B^{-1}\left(\frac{1}{s}\right). \end{aligned}$$

$$\|h_1+h_2+h_3\|_\infty \leqq 2C^{-1}\left(\frac{1}{s}\right)+sf^*(s)B^{-1}\left(\frac{1}{s}\right)+\int_{f^*(s)}^\infty m(f,y)\,dy\cdot B^{-1}\left(\frac{1}{s}\right)$$

$$=2C^{-1}\left(\frac{1}{s}\right)+\left[sf^*(s)+\int_{f^*(s)}^\infty m(f,y)\,dy\right]B^{-1}\left(\frac{1}{s}\right)$$

$$=2C^{-1}\left(\frac{1}{s}\right)+sf^{**}(s)B^{-1}\left(\frac{1}{s}\right)$$

$$\leqq 2C^{-1}\left(\frac{1}{s}\right)+sA^{-1}\left(\frac{1}{s}\right)B^{-1}\left(\frac{1}{s}\right)\leqq 3C^{-1}\left(\frac{1}{s}\right).$$

$$\|h_4\|_1=\|T(f^u,g^w)\|_1\leqq\|f^u\|_1\|g^w\|_1$$

$$=\int_u^\infty m(f,y)\,dy\cdot\int_w^\infty m(g,y)\,dy=\int_{f^*(s)}^\infty m(f,y)\,dy\cdot\int_w^\infty m(g,y)\,dy$$

$$\leqq sf^{**}(s)\int_w^\infty m(g,y)\,dy\leqq sA^{-1}\left(\frac{1}{s}\right)\int_w^\infty m(g,y)\,dy.$$

$$m(h_4,t)\leqq\frac{\|h_4\|_1}{t}\leqq\frac{sA^{-1}\left(\frac{1}{s}\right)}{t}\int_w^\infty m(g,y)\,dy.$$

But

$$m\left(h,3C^{-1}\left(\frac{1}{s}\right)+\theta C^{-1}\left(\frac{1}{s}\right)\right)\leqq m\left(h_4,\theta C^{-1}\left(\frac{1}{s}\right)\right)$$

$$\leqq\frac{sA^{-1}\left(\frac{1}{s}\right)}{\theta C^{-1}\left(\frac{1}{s}\right)}\int_{B^{-1}(1/s)}^\infty m(g,y)\,dy.$$

Let $z=C^{-1}(1/s)$; then

$$s=\frac{1}{C(z)},\quad\frac{sA^{-1}\left(\frac{1}{s}\right)}{C^{-1}\left(\frac{1}{s}\right)}\leqq\frac{1}{B^{-1}\left(\frac{1}{s}\right)}=\frac{1}{B^{-1}(C(z))}.$$

$$m\left(\frac{h}{3+\theta},z\right)=m(h,(3+\theta)z)\leqq m\left(h_4,\theta C^{-1}\left(\frac{1}{s}\right)\right)$$

$$\leqq\frac{sA^{-1}\left(\frac{1}{s}\right)}{\theta C^{-1}\left(\frac{1}{s}\right)}\int_{B^{-1}(1/s)}^\infty m(g,y)\,dy\leqq\frac{1}{\theta B^{-1}(C(z))}\int_{B^{-1}(C(z))}^\infty m(g,y)\,dy.$$

Lemma 4.2 is established.

Before stating our principal theorem we need a fact about Young's functions. Orlicz [6] considered Young's functions which satisfy the inequality $A(2x) \leqq KA(x)$. It can be shown [3, p. 26, Lemma 4.1] that a Young's function satisfies Orlicz' condition if and only if there is a constant $p \geqq 1$ such that

$$xa(x) \leqq pA(x) \tag{4.3}$$

where

$$\int_0^x a(t)\,dt = A(x).$$

(4.3) with $p < \infty$ is equivalent to a condition on the complementary Young's function B, namely

$$xb(x) \geqq p'B(x), \qquad p' > 1, \tag{4.4}$$

where $1/p + 1/p' = 1, b(x)$ is inverse to $a(x)$ and $B(x) = \int_0^x b(t)\,dt$. A Young's function B which satisfies (4.4) gives rise to an Orlicz space L_B which in some sense is bounded away from L and thus avoids some of the well-known pathology of the latter space. We shall need one fact about a Young's function satisfying (4.4).

Lemma 4.5. *If B is a Young's function such that*

$$pB(x) \leqq xb(x), \qquad p > 1,$$

then

$$\int_0^x \frac{b(t)}{t}\,dt \leqq p'\frac{B(x)}{x} \leqq \frac{p'}{p}\,b(x),$$

where $1/p + 1/p' = 1$.

Proof. Integrate by parts.

$$I = \int_0^x \frac{b(t)}{t}\,dt = \frac{B(t)}{t}\Big|_0^x + \int_0^x \frac{B(t)}{t^2}\,dt$$

$$\leqq \frac{B(x)}{x} - \lim_{t\to 0}\frac{B(t)}{t} + \frac{1}{p}\int_0^x \frac{tb(t)}{t^2}\,dt \leqq \frac{B(x)}{x} + \frac{1}{p}I.$$

$$\frac{1}{p'}I \leqq \frac{B(x)}{x} \leqq \frac{b(x)}{p}.$$

Corollary 4.6. *If B is a Young's function such that*

$$pB(x) \leqq xb(x), \qquad p > 1,$$

then

$$\int_0^{B(y)} \frac{dt}{B^{-1}(t)} \leqq p'b(y).$$

Proof. Let $t = B(u)$.

$$\int_0^{B(y)} \frac{dt}{B^{-1}(t)} = \int_0^{y} \frac{b(u)}{u} \leqq \frac{p'}{p}\, b(y) \leqq p'b(y).$$

THEOREM 4.7 (FRACTIONAL INTEGRATION FOR ORLICZ SPACES). *If A, B are Young's functions such that*

$$\int_0^1 \frac{A^{-1}(t)B^{-1}(t)}{t^2}\, dt < \infty,$$

$$pB(x) \leqq xb(x), \qquad p > 1,$$

and the Young's function C is defined by

$$C^{-1}(x) = \int_0^x \frac{A^{-1}(t)B^{-1}(t)}{t^2}\, dt,$$

and if, furthermore, $f \in M_A$, $g \in L_B$ and $h = T(f,g)$ where T is a convolution operator, then $h \in L_C$ and

$$\|h\|_C \leqq 4p' \|f\|_{M_A} \|g\|_B.$$

Proof. By the homogeneity of T it will be enough if we show that for $\|f\|_{M_A} \leqq 1$, $\|g\|_B < 1$ we have

$$\int C\left(\frac{|h(x)|}{4p'}\right) d\nu < 1.$$

We note that $3 + p' \leqq 4p'$ and so by Lemma 4.2 with $\theta = p'$,

$$\begin{aligned}
\int C\left(\frac{|h(x)|}{4p'}\right) d\nu &\leqq \int C\left(\frac{|h(x)|}{3+p'}\right) d\nu \\
&= \int_0^\infty m\left(\frac{h}{3+p'}, z\right) c(z)\, dz \\
&\leqq \int_0^\infty c(z) \frac{1}{p'B^{-1}(C(z))} \int_{B^{-1}(C(z))}^\infty m(g,y)\, dy \\
&= \frac{1}{p'} \int_0^\infty m(g,y)\, dy \int_0^{C^{-1}(B(y))} \frac{c(z)}{B^{-1}(C(z))}\, dz
\end{aligned}$$

(let $t = C(z)$ and use Corollary 4.6)

$$\begin{aligned}
&= \frac{1}{p'} \int_0^\infty m(g,y)\, dy \int_0^{B(y)} \frac{dt}{B^{-1}(t)} \leqq \frac{1}{p'} \int_0^\infty m(g,y) p'b(y)\, dy \\
&= \int B(|g(x)|)\, d\bar{\mu} < 1. \qquad \text{Q.E.D.}
\end{aligned}$$

Several known theorems on L^p spaces are implied by Theorem 4.7. We shall demonstrate only the two most celebrated of these.

THEOREM OF HARDY AND LITTLEWOOD [2, *p.* 595, *Theorem* 4]. *If* $g \in L^p$, $p > 1$, *and* $0 < \alpha < 1/p$ *then* g_α, α*th fractional integral of* g, *belongs to* L^r *where*

$$\frac{1}{r} = \frac{1}{p} - \alpha.$$

Proof. By definition, $g_\alpha(x) = \int_0^\infty g(x-t)F_\alpha(t)\,dt$ where $F_\alpha(t) = 1/\Gamma(\alpha)t^{1-\alpha}$. But clearly $F_\alpha(t) \in M_A$ where $A(x) = x^{1/(1-\alpha)}$ and $g \in L_B$ where $B(x) = x^p$. Then $b(x) = px^{p-1}$ so that $pB(x) \leqq xb(x)$ with $p > 1$.

$$C^{-1}(x) = \int_0^x \frac{t^{1-\alpha}t^{1/p}}{t^2}\,dt = \frac{x^{1/r}}{1/r}.$$

Therefore $L_C = L^r$ and the theorem is established.

THEOREM OF SOBOLEFF [7, *p.* 481, *Theorem*]. *If* $g \in L^p$ *on n-space and* $0 < \alpha < n/p$ *then* g_α, *the* α*th fractional integral of* g, *belongs to* L^r *where* $1/r = 1/p - \alpha/n$.

Proof. By definition $g_\alpha(X) = \int g(X-T)F_\alpha(T)\,dT$, where $X = (x_1, \cdots, x_n)$, $T = (t_1, \cdots, t_n)$, $dT = dt_1 \cdots dt_n$ and $F_\alpha(T)$ is of the form $K_{n,\alpha}/|T|^{n-\alpha}$. By integrating $F_\alpha(T)$ over spheres centered at the origin it is easily seen that $F_\alpha \in M_A$, where $A(x) = x^{n/(n-\alpha)}$.

$$C^{-1}(x) = \int_0^x \frac{t^{1-\alpha/n}t^{1/p}}{t^2}\,dt = \frac{x^{1/r}}{1/r}.$$

Thus $L_C = L^r$.

V. **Endpoint results for fractional integration.** If we wish to form the convolution of a function $f \in M_A$ with a function $g \in L_B$ and $B(x)$ fails to satisfy condition (4.4) then we need to complicate slightly the previous analysis. We first state a more delicate form of Lemma 4.2 which leads to our result.

LEMMA 5.1. *Under the hypotheses and notation of Lemma* 4.2 *the following conclusion holds*:

$$\int_{3C^{-1}(z)}^\infty m(h,y)\,dy \leqq \frac{A^{-1}(z)}{z}\int_{B^{-1}(z)}^\infty m(g,y)\,dy.$$

The proof proceeds the same as the proof of Lemma 4.2 up to the point where we conclude that

$$\|h_4\|_1 \leqq sA^{-1}\left(\frac{1}{s}\right)\int_w^\infty m(g,y)\,dy.$$

But since $h = h_1 + h_2 + h_3 + h_4$ and $\|h_1 + h_2 + h_3\|_\infty \leqq 3C^{-1}(1/s)$ it follows that

$$\int_{3C^{-1}(1/s)}^{\infty} m(h,y)\,dy \leqq \|h_4\|_1 \leqq sA^{-1}\left(\frac{1}{s}\right) \int_w^\infty m(g,y)\,dy.$$

But $w = B^{-1}(1/s)$, and the conclusion follows by setting $z = 1/s$.

THEOREM 5.2. *If A, B are Young's functions such that*

$$\int_0^1 \frac{A^{-1}(t)B^{-1}(t)}{t^2}\,dt < \infty,$$

if the Young's function C is defined by

$$C^{-1}(x) = \int_0^x \frac{A^{-1}(t)B^{-1}(t)}{t^2}\,dt,$$

if $r(x)$ is any non-negative, nondecreasing function on the non-negative reals such that

$$r(x) \leqq \int_0^{C(x)} \frac{t}{A^{-1}(t)}\,db(B^{-1}(t)),$$

$(b(x) = dB(x)/dx)$, and if

$$R(x) = \int_0^x r(t)\,dt,$$

then, if $f \in M_A$, $g \in L_B$ and $h = T(f,g)$ where T is a convolution operator then $h \in L_R$ and

$$\|h\|_R \leqq 3\|f\|_{M_A}\|g\|_B.$$

Proof. Assume that $\|f\|_{M_A} \leqq 1$, $\|g\|_B < 1$. Let

$$V(x) = \int_0^x \frac{t}{A^{-1}(t)}\,db(B^{-1}(t)).$$

If we form the Stieltjes integral of both sides of the equation of Lemma 5.1 with respect to $dV(z) = (z/A^{-1}(z))db(B^{-1}(z))$, then

$$\int_0^\infty dV(z) \int_{3C^{-1}(z)}^\infty m(h,y)\,dy \leqq \int_0^\infty \frac{A^{-1}(z)}{z}\,dV(z) \int_{B^{-1}(z)}^\infty m(g,y)\,dy$$

$$= \int_0^\infty db(B^{-1}(z)) \int_{B^{-1}(z)}^\infty m(g,y)\,dy = \int_0^\infty db(x) \int_x^\infty m(g,y)\,dy$$

$$= \int_0^\infty m(g,y)b(y)\,dy < 1.$$

But

$$\int R\left(\frac{|h|}{3}\right) dv = \int_0^\infty r(x) m\left(\frac{h}{3}, x\right) dx$$

$$= \int_0^\infty m(h, 3x) r(x)\, dx \leqq \int_0^\infty m(h, 3x)\, dx \int_0^{C(x)} \frac{t}{A^{-1}(t)}\, db(B^{-1}(t))$$

$$= \int_0^\infty m(h, 3x) dx \int_0^{C(x)} dV(z)$$

(let $3x = y$)

$$= \frac{1}{3}\int_0^\infty m(h, y)\, dy \int_0^{C(y/3)} dV(z)$$

$$= \frac{1}{3}\int_0^\infty dV(z) \int_{3C^{-1}(z)}^\infty m(h, y)\, dy < \frac{1}{3} < 1.$$

Thus

$$||h||_R \leqq 3$$

and the general conclusion of Theorem 5.2 follows by homogeneity of the norms.

We shall give an application of Theorem 5.2. $M(\log M)^\alpha, \alpha > 0$ denotes M_A on a measure space of finite total measure, where $A(x) \sim x(\log x)^\alpha$. $F(x) \sim G(x)$ means $\lim_{x\to\infty} F(x)/G(x) = 1$. To simplify the discussion we shall use convolution on $(0, 1)$ rather than a general convolution operator.

Theorem 5.3. *If $f \in M(\log M)^\alpha, \alpha > 0$ and $g \in L(\log L)^\beta$, $\beta > 0$, on $(0, 1)$ are extended periodically then their convolution $h(x) = \int_0^1 f(x-t)g(t)\, dt$ is in $L(\log L)^{\alpha+\beta}$ on $(0, 1)$. (More precisely, $h \in \beta L(\log L)^{\alpha+\beta}$.)*

Proof. The following remarks should enable the reader to do the necessary computations himself:

1°. The properties of the Orlicz space L_A over a totally finite measure space depend only on the values of $A(x)$ for x sufficiently large. Thus the integrals occuring in the statement of Theorem 5.2 may be taken with any convenient lower limit of integration.

2°. If $A(x) \sim x(\log x)^\alpha$, then $A^{-1}(x) \sim x(\log x)^{-\alpha}$.

We then find using the notation of Theorem 5.2: $A(x) \sim x(\log x)^\alpha$, $B(x) \sim x(\log x)^\beta$, $C(x) \sim x(\log x)^{\alpha+\beta}$ and $R(x) \sim \beta x(\log x)^{\alpha+\beta}$. By Theorem 5.2, $h \in L_R = \beta L(\log L)^{\alpha+\beta} = \beta L_C = L_C$. This proves Theorem 5.3.

Theorem 5.3 generalizes the following theorem of Zygmund [9, p. 609, Corollary].

Theorem. *If $f \in L(\log L)^\alpha, \alpha > 0, g \in L(\log L)^\beta, \beta > 0$, on (0.1) are extended periodically then their convolution belongs to $L(\log L)^{\alpha+\beta}$.*

It is to be remarked that, of course, the above theorem also follows immediately from Theorem 2.5.

It is interesting to compare Theorem 2.5 with Theorem 5.2. We shall restrict ourselves to ordinary convolution on the interval (0,1). Theorem 2.5 says that if $f \in L_A, g \in L_B$ and

$$A^{-1}(x)B^{-1}(x) = O(xC^{-1}(x)), \tag{5.4}$$

then in all cases $f * g \in L_C$. Theorem 5.2 says that if $f \in M_A$, $g \in L_B$ then in certain cases the condition (5.4) is sufficient to give $f * g \in L_C$ and in certain other cases not. For example if $f \in M^p = M_A, p > 1$ and $g \in L(\log L)^{1/p} = L_B$ then there follows from Theorem 5.2 the well-known fact [9, p. 606 Theorem 2] that $f * g \in L^p = L_R$ and not $L^p \log L = L_C$.

Theorem 5.2 seems to say that if we start with a function $g \in L_B$ and form the convolution with a function $f \in M_A$ then A must not grow too much faster than B if (5.4) is to guarantee that $f * g \in L_C$.

Example. If $f \in M(\log\log M)^\alpha, \alpha > 0$, and $g \in L(\log L)^\beta, \beta > 0$ then (5.4) is sufficient to give us that $f * g \in L_C = L(\log L)^\beta(\log\log L)^\alpha$. If, however, the situation is reversed and $f \in M(\log M)^\beta$ while $g \in (\log\log L)^\alpha$ then (5.4) is not sufficient and indeed

$$f * g \in L_R = L(\log L)^\beta(\log\log L)^{\alpha-1} \supsetneqq L(\log L)^\beta(\log\log L)^\alpha = L_C.$$

VI. **Further remarks on the generalized Hölder's inequality.** Our first theorem completes Lemma 2.1.

Theorem 6.1. *If $A(x), B(x), C(x)$ are nondecreasing functions from $[0, \infty]$ into $[0, \infty]$ which are continuous to the left, if $A(0) = B(0) = C(0) = 0$, if A^{-1}, B^{-1}, C^{-1} are their inverses normalized so as to be continuous to the right and if $A^{-1}(\infty) = B^{-1}(\infty) = C^{-1}(\infty) = \infty$ then for all $x \geqq 0$,*

$$A(A^{-1}(x)) \leqq x \leqq A^{-1}(A(x)).$$

Similarly for B and B^{-1}, C and C^{-1}. If there exists a number $K > 0$, such that for all $x \geqq 0, A^{-1}(x)B^{-1}(x) \leqq KC^{-1}(x)$ then for all $x \geqq 0, y \geqq 0$,

$$C\left(\frac{xy}{K}\right) \leqq A(x) + B(y).$$

Conversely, if for all $x \geqq 0, y \geqq 0, C(xy/K) \leqq A(x) + B(y)$, then for all $x \geqq 0$,

$$A^{-1}(x)B^{-1}(x) \leqq KC^{-1}(2x) \leqq 2KC^{-1}(x).$$

Remark. A and A^{-1} are related by the following equations:

$$A^{-1}(y) = \inf\{x : A(x) > y\},$$

$$A(x) = \sup\{y : A^{-1}(y) < x\},$$

where $\sup \emptyset = 0, \inf \emptyset = \infty$.

Proof. The first part of the theorem is established in exactly the same way as Lemma 2.1. To prove the last statement for any given $u \geqq 0$ we let $x = A^{-1}(u), y = B^{-1}(u)$.

$$C\left(\frac{xy}{K}\right) \leqq A(x) + B(y),$$

$$C\left(\frac{A^{-1}(u)B^{-1}(u)}{K}\right) \leqq A(A^{-1}(u)) + B(B^{-1}(u)) \leqq 2u.$$

$$\frac{A^{-1}(u)B^{-1}(u)}{K} \leqq C^{-1}\left(C\left(\frac{A^{-1}(u)B^{-1}(u)}{K}\right)\right) \leqq C^{-1}(2u).$$

The theorem is established.

Two closely related problems immediately suggest themselves.

PROBLEM 6.2. *Given two Young's functions A, B how shall we choose a Young's function C such that whenever $f \in L_A$ and $g \in L_B$ then $fg \in L_C$?*

PROBLEM 6.3. *Given two Young's functions A and C, how shall we choose a Young's function B such that whenever $f \in L_A$ and $g \in L_B$ then* $fg \in L_C$?

To solve Problem 6.2 we consider a subset of the first quadrant.

$$E = \{(x,y) : x \geqq 0, 0 \leqq y \leqq A^{-1}(x)B^{-1}(x)\}.$$

Let $\bar{E} =$ convex closure of E. If $\bar{E} \neq$ first quadrant, it is bounded above by a concave curve which we denote by $y = C^{-1}(x)$. Clearly $A^{-1}(x)B^{-1}(x) \leqq C^{-1}(x)$ so that by Theorem 2.3, $f \in L_A$ and $g \in L_B$ implies $fg \in L_C$

THEOREM 6.4. *If $\bar{E} =$ first quadrant then there exist functions $f \in L_A(0,1)$, $g \in L_B(0,1)$ such that the product $h(x) = f(x)g(x)$ does not belong to any Orlicz space.*

Sketch of proof. $\bar{E} =$ first quadrant if and only if

$$\limsup_{x \to \infty} \frac{A^{-1}(x)B^{-1}(x)}{x} = \infty.$$

Let $1 < y_1 < y_2 < y_3 \cdots$ such that $A^{-1}(y_n)B^{-1}(y_n) > 2^n y_n$. Let $x_n = 1/2^n y^n$. Let

$$r(x) = \begin{cases} 0 & \text{if } \sum_{j=1}^{\infty} x_j < x, \\ y_n & \text{if } \sum_{j=n+1}^{\infty} x_j < x \leqq \sum_{j=n}^{\infty} x_j. \end{cases}$$

Let $f(x) = A^{-1}(r(x)), g(x) = B^{-1}(r(x))$. The reader may easily show that

$$\int_0^1 r(x)\,dx = 1 \quad \text{so that } \|f\|_A = 1, \quad \|g\|_B = 1.$$

But $\int_0^1 f(x)g(x)\,dx = \infty$.

Thus $f(x)g(x)$ is not integrable in any neighborhood $(0,\epsilon)$ of the origin and so cannot belong to any Orlicz space.

Theorem 6.5. *Let A, B, C be Young's functions. The following conditions are equivalent*:

1. $\limsup_{x\to\infty} A^{-1}(x)B^{-1}(x)/C^{-1}(x) < \infty$.
2. *There exist numbers $K > 0, x_0 \geqq 0$ such that for all $x, y \geqq x_0$,*

$$C\left(\frac{xy}{K}\right) \leqq A(x) + B(y).$$

3. *There is a number $M > 0$ such that for all measurable functions f, g on* $(0,1)$,

$$\|fg\|_C \leqq M\|f\|_A\|g\|_B.$$

4. *For every $f \in L_A(0,1)$ and $g \in L_B(0,1)$, fg belongs to $L_C(0,1)$.*

Remark. The equivalence of 3 and 4 appears in Krasnosel'skii and Rutickii [3, Lemma 13.5, p. 118]. The equivalence of 2 and 4 is due to Andô [1, Theorem 1, p. 178].

Sketch of proof. 1 implies that there exist numbers $u_0 > 0$ and $K > 0$ such that for $x > u_0$, $A^{-1}(x)B^{-1}(x) < KC^{-1}(x)$. Let

$$x_0 = \max(A^{-1}(u_0), B^{-1}(u_0)).$$

For $x, y > x_0$ we may reason as in Lemma 2.1 to establish 2. Thus 1 implies 2.

If 2 is satisfied let $M = KL$ where $L = 2 + A(x_0) + B(x_0)$. Suppose without loss of generality that $\|f\|_A < 1, \|g\|_B < 1$. Let $F = \{x : f(x) \geqq x_0\}$, $G = \{x : g(x) \geqq x_0\}$.

$$\begin{aligned}
&\int_0^1 C\left(\frac{|f(x)g(x)|}{M}\right)dx \\
&\leqq \frac{1}{L}\int_0^1 C\left(\frac{|f(x)g(x)|}{K}\right)dx \\
&\leqq \frac{1}{L}\left(\int_{F\cap G} + \int_{F-G} C\left(\frac{|f(x)|x_0}{K}\right)dx + \int_{G-F} C\left(\frac{x_0|g(x)|}{K}\right)dx\right. \\
&\qquad\qquad \left. + \int_{(F\cup G)'} C\left(\frac{x_0^2}{K}\right)dx\right) \\
&\leqq \frac{1}{L}\left(\int_F A(|f(x)|)\,dx + \int_{F'} A(x_0)\,dx + \int_G B(|g(x)|)\,dx + \int_{G'} B(x_0)\,dx\right) \\
&< \frac{1}{L}(1 + A(x_0) + 1 + B(x_0)) = 1.
\end{aligned}$$

Observe that we have used strongly the fact that $(0,1)$ is a totally finite measure space. We have shown that 2 implies 3. 3 obviously implies 4.

To show that 4 implies 1 we assume 1 false. Then reasoning as in the proof of Theorem 6.4, we find $1 < y_1 < y_2 < \cdots$ such that $A^{-1}(y_n)B^{-1}(y_n) > 2^n C^{-1}(y_n)$. Starting with the sequence $y_1, y_2, \cdots$, we define $r(x), f(x), g(x)$ as in Theorem 6.4; then $\|f\|_A = 1, \|g\|_B = 1$. But for any $\theta > 0$, there is an integer n such that $2^n\theta > 1$. Therefore,

$$\int_0^1 C(\theta f(x)g(x))\,dx \geqq \sum_{j=n+1}^{\infty} C(\theta A^{-1}(y_j)B^{-1}(y_j))x_j$$

$$\geqq \sum_{j=n+1}^{\infty} C(\theta 2^j C^{-1}(y_j))x_j \geqq \theta \sum_{j=n}^{\infty} 2^{j-1} C(2C^{-1}(y_j))x_j = \infty .$$

(The reader may verify for himself that $C(2C^{-1}(y_n)) \geqq y_n$.)

THEOREM 6.6. *Let A, B, C be Young's functions. The following conditions are equivalent:*

1. $\limsup_{x\to 0^+} A^{-1}(x)B^{-1}(x)/C^{-1}(x) < \infty$.
2. *There exist numbers $K > 0, x_0 > 0$ such that for all $x, y \leqq x_0$,*

$$C\left(\frac{xy}{K}\right) \leqq A(x) + B(y).$$

3. *There is a number M such that for all sequences $f = (f_1, f_2, \cdots), g = (g_1, g_2, \cdots)$,*

$$\|fg\|_C \leqq M\|f\|_A\|g\|_B.$$

4. *For every sequence $f = (f_1, f_2, \cdots)$ in L_A and every sequence $g = (g_1, g_2, \cdots)$ in L_B, the product sequence $fg = (f_1g_1, f_2g_2, \cdots)$ belongs to L_C.*

Sketch of proof. The proof is analogous to the proof of Theorem 6.5. The only point that might cause the reader difficulty is in showing that 4 implies 1.

If 1 is false then we may choose inductively a sequence y_n.

$$\frac{1}{4} > y_1, \qquad A^{-1}(y_1)B^{-1}(y_1) > 2C^{-1}(y_1).$$

$$y_n < \frac{1}{2}y_{n-1}, \qquad A^{-1}(y_n)B^{-1}(y_n) > 2^n C^{-1}(y_n).$$

Let m_n be the unique integer such that

$$\frac{1}{2^{n+1}y_n} \leqq m_n < \frac{1}{2^{n+1}y_n} + 1.$$

But $1 < 1/2^{n+1}y_n$ so that $m_n < 1/2^n y_n$. Let $r_j = y_n$ if $\sum_{k=1}^{n-1} m_k \leqq j < \sum_{k=1}^{n} m_k$. Then

$$\sum_{j=1}^{\infty} r_j = \sum_{n=1}^{\infty} m_n y_n < \sum_{n=1}^{\infty} \frac{1}{2^n y_n} y_n = 1.$$

Let $f_j = A^{-1}(r_j), g_j = B^{-1}(r_j)$, then $\|f\|_A < 1, \|g\|_B < 1$. But given any $\theta > 0$ there is an integer N such that $2^N\theta > 1$. Therefore

$$\begin{aligned}\sum_{j=1}^{\infty} C(\theta f_j g_j) &= \sum_{n=1}^{\infty} m_n C(\theta A^{-1}(y_n) B^{-1}(y_n)) \\ &\geqq \sum_{n=1}^{\infty} m_n C(\theta 2^n C^{-1}(y_n)) \geqq \sum_{n=N+1}^{\infty} m_n C(\theta 2^n C^{-1}(y_n)) \\ &\geqq \sum_{n=N+1}^{\infty} m_n 2^{n-1}\theta C(2C^{-1}(y_n)) = \infty.\end{aligned}$$

We may combine Theorem 6.5 and Theorem 6.6 to obtain

THEOREM 6.7. *Let A, B, C be Young's functions. The following conditions are equivalent*:

1. *There is a number $K > 0$ such that for all $x \geqq 0, A^{-1}(x)B^{-1}(x) \leqq KC^{-1}(x)$.*
2. *There is a number $K > 0$ such that for all $x, y \geqq 0$,*

$$C\left(\frac{xy}{K}\right) \leqq A(x) + B(y).$$

3. *There is a number $M > 0$ such that, for all measure spaces (X, μ), if f and g are measurable then*

$$\|fg\|_C \leqq M\|f\|_A\|g\|_B.$$

4. *For each measure space (X, μ), if $f \in L_A$ and $g \in L_B$ then $fg \in L_C$.*

The following theorem which together with Theorem 6.4 solves Problem 6.2 follows as an immediate corollary of Theorem 6.7. We therefore leave the proof to the reader.

THEOREM 6.8. *If A, B are Young's functions such that $\bar{E}$ = convex closure of $\{(x,y) : x \geqq 0, 0 \leqq y \leqq A^{-1}(x)B^{-1}(x)\}$ is not the entire first quadrant and if we denote by $y = C^{-1}(x)$ the upper boundary of $\bar{E}$, then A, B, D are Young's functions such that $f \in L_A(X)$, $g \in L_B(X)$ implies, for any measure space X, that their product $fg \in L_D(X)$ if and only if there is a positive constant K such that $D(x) \leqq C(Kx)$. In this case it follows that*

$$\|fg\|_D \leqq 2K\|f\|_A\|g\|_B.$$

We now consider Problem 6.3. This was solved by Andô [1, Theorem 4] and [5, pp. 180-181]. We may rephrase his theorem in our terminology. (He considered N-functions which is a less general notion than that of Young's function.)

THEOREM OF ANDÔ. *If A and C are Young's functions and for $y \geqq 0$,*

$$B(y) = \sup_{x \geqq 0} [C(xy) - A(x)], \tag{6.9}$$

then B is a Young's function.

A, D, C are Young's functions such that for any measure space $X, f \in L_A(X)$, $g \in L_D(X)$ implies that their product $fg \in L_C(X)$ if and only if there is a positive constant K such that $D(x) \geqq B(x/K)$.

REMARK. B may be the "trivial" Young's function.

We close with a theorem which throws some light on the "relative complementation" defined by equation (6.9).

THEOREM. *If A and C are Young's functions and for $x, y \geqq 0$,*

$$B(y) = \sup_{x \geqq 0} (C(xy) - A(x)),$$

$$A_1(x) = \sup_{y \geqq 0} (C(xy) - B(y)),$$

$$B_1(y) = \sup_{x \geqq 0} (C(xy) - A_1(x)),$$

then $A_1(x) \leqq A(x)$ but $B_1(y) = B(y)$.

The proof is easy and we leave it as an exercise for the reader. That the inequality $A_1(x) \leqq A(x)$ may not be replaced by equality is easily seen by means of an example.

Let

$$C(x) = \begin{cases} 0 & \text{if } 0 \leqq x \leqq 1, \\ x - 1 & \text{if } x > 1, \end{cases}$$

and let $A(x) = x^2$. Then $A_1(1/2) = 0 < 1/4 = A(1/2)$.

VII. Further remarks on the generalized Young's theorem.

CONVERSE TO LEMMA 2.4. *If A, B, C are Young's functions such that for all $x \geqq 0, y \geqq 0$,*

$$xy \leqq A(x)C^{-1}(B(y)) + B(y)C^{-1}(A(x)),$$

then for any $u \geqq 0$,

$$A^{-1}(u)B^{-1}(u) \leqq 2uC^{-1}(u).$$

Proof. Let $x = A^{-1}(u), y = B^{-1}(u)$. Q.E.D.

If we restrict ourselves to Orlicz spaces of a special form (namely L^p spaces) then the results of Theorem 2.3, Theorem 2.5 and Theorem 4.7 may be seen to be best possible by various methods. But if we consider function spaces more general than the Orlicz spaces then this is no longer true as far as Theorem 2.5 and Theorem 4.7 are concerned. The reader who wishes to understand this point should refer to a paper of the author [5].

We shall discuss here an improvement in Theorem 2.5 which follows as a consequence of Theorem 4.7.

We need for this purpose the "associated space" of the space M_A. Like M_A this was introduced by G.G. Lorentz [4].

DEFINITION 7.1. *Let A and $\bar{A}$ be complementary Young's functions. Let f be a function on a measure space (X,μ).*

$$||f||_{K_A} = \sup \int f(x)g(x)\, d\mu,$$

sup *over all $g \in M_{\bar{A}}$ of norm less than or equal to* 1.

$$K_A = \{f : ||f||_{K_A} < \infty\}.$$

LEMMA 7.2. $||f||_{M_A} \leqq ||f||_A \leqq ||f||_{K_A}$.

Proof. The first inequality is Lemma 3.1. It is proved in the standard treatises on Orlicz spaces [3, p. 80] or [10, Vol. I, pp. 170-175] that

$$||f||_A \leqq \sup_g \int f(x)g(x)\, d\mu$$

sup over g such that $||g||_{\bar{A}} \leqq 1$. But $||g||_{M_{\bar{A}}} \leqq ||g||_{\bar{A}}$ so that the sup above is less than or equal to the sup taken over $g \in M_{\bar{A}}$ with $||g||_{M_{\bar{A}}} \leqq 1$. Therefore $||f||_A \leqq ||f||_{K_A}$.

THEOREM 7.3. *If A, B, C are Young's functions such that for all $x > 0$,*

1. *$pA(x) \leqq xA'(x)$ for a given $p > 1$,*
2. *$qB(x) \leqq xB'(x)$ for a given $q > 1$,*
3. *$A^{-1}(x)B^{-1}(x) \leqq xC^{-1}(x)$,*

and if $f \in L_A$, $g \in L_B$ on a locally compact unimodular topological group G, then their convolution

$$h(x) = \int f(t)g(t^{-1}x)\, dt$$

is in K_C and

$$||h||_{K_C} \leqq 16p'q'\, ||f||_A ||g||_B,$$

where $1/p + 1/p' = 1$, $1/q + 1/q' = 1$.

Proof. Suppose $v \in M_{\bar{C}}$, $||v||_{M_{\bar{C}}} \leqq 1$.

$$\begin{aligned}\int_G v(x)h(x)\,dx &\leqq \int |v(x)| \int |f(t)||g(t^{-1}x)|\,dtdx \\ &= \int |f(t)| \int |v(x)||g(t^{-1}x)|\,dxdt \\ &= \int |f(t)|w(t)\,dt.\end{aligned}$$

$w(t) = \int |v(x)||g(t^{-1}x)|\,dx$ is the convolution of a function in $M_{\bar{C}}$ of norm less than or equal to 1 with a function in L_B, and so by Theorem 4.7 is in L_D, $||w||_D \leqq 4q'||g||_B$, where

$$\begin{aligned}D^{-1}(x) &= \int_0^x \frac{\bar{C}^{-1}(t)B^{-1}(t)}{t^2}\,dt \\ &\leqq \int_0^x \frac{2t}{C^{-1}(t)}\,\frac{B^{-1}(t)}{t^2}\,dt \leqq 2\int_0^x \frac{dt}{A^{-1}(t)} \\ &= 2\int_0^{A^{-1}(x)} \frac{A'(u)}{u}\,du \leqq 2p'\,\frac{A(u)}{u}\,\Bigg|_0^{A^{-1}(x)} \\ &\leqq 2p'\,\frac{x}{A^{-1}(x)}.\end{aligned}$$

In the above evaluation we have used Property 1.6 and Lemma 4.5.

$$\begin{aligned}A^{-1}(x)D^{-1}(x) &\leqq 2p'x \leqq A^{-1}(x)\bar{A}^{-1}(x), \\ D^{-1}(x) &\leqq 2p'\bar{A}^{-1}(x), \\ \bar{A}\left(\frac{x}{2p'}\right) &\leqq D(x)\end{aligned}$$

so that

$$\left\|\frac{w}{2p'}\right\|_{\bar{A}} \leqq ||w||_D.$$

We use equation (1.7).

$$\begin{aligned}\int v(x)h(x)\,dx &\leqq \int |f(t)|w(t)\,dt \\ &\leqq 2||f||_A||w||_{\bar{A}} \leqq 4p'||f||_A||w||_D \\ &\leqq 16p'q'||f||_A||g||_B.\end{aligned}$$

But $||h||_{K_C} = \sup_v \int v(x)h(x)dx$ where $||v||_{M_{\bar{C}}} \leqq 1$.

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